$$\alpha(\tau) = (f_2^2(\tau) + g_2^2(\tau))^{1/2}, \quad \beta(\tau) = (f_1^2(\tau) + g_1^2(\tau))^{1/2}$$

$$f_i(\tau) = \int_0^\tau e^{-k_i t} \cos \omega_i t\, dt, \quad g_i(\tau) = \int_0^\tau e^{-k_i t} \sin \omega_i t\, dt, \quad i = 1, 2$$

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CONTROLLABILITY OF A NONLINEAR SYSTEM IN A LINEAR APPROXIMATION

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We study the conditions for the controllability of a dynamic system whose behavior in a finite-dimensional phase space is described by a nonlinear differential equation. The results obtained complement the investigations in [1-10].

1. Definitions and formulations of results. Let R^n be an n-dimensional arithmetic space of points $x = \operatorname{col}(x_1, \ldots, x_n)$ with norm $|\cdot|$. We examine the system

$$x^{\cdot} = A(t)x + B(t)u + \varphi(t, x, u), \; x \in R^n, \; u \in R^m, \; t \in [t_0, \infty) \tag{1.1}$$

Here the real ($n \times n$) and ($n \times m$) matrices $A(t)$ and $B(t)$ are continuous for $t \in [t_0, \infty)$; the real function $\varphi(t, x, u)$ is continuous in the collection of arguments $(t, x, u) \in [t_0, \infty) \times R^n \times R^m$. We say that the control $u_0(t)$, $t \in I = [t_0, t_1]$ translates the position (t_0, x_0) of system (1.1) into the position $(t_1, 0)$ if the solution $x_0(t)$, satisfying the initial condition $x(t_0) = x_0$ of system (1.1) under control $u = u_0(t)$ is defined for all $t \in I$, is unique on I, and passes through the point $x_1 = 0$ at instant $t_1 : x_0(t_1) = 0$.

Without further stipulation we shall assume everywhere below that the following two conditions are fulfilled.

1) The linear system

$$x^{\cdot} = A(t)x + B(t)u \tag{1.2}$$

is completely controllable on a fixed time interval $I = [t_0, t_1]$. This signifies that for any $x_0 \in R^n$ there exists a control $u_0(t)$, continuous on I, which translates the position (t_0, x_0) of system (1.2) into the position $(t_1, 0)$.

2) System (1.1) possesses the property of right-hand uniqueness, i.e. for any $x_0 \in R^n$ and any control $u_0(t)$ continuous on I, the solution of the Cauchy problem for

$$x^{\cdot} = A(t)x + B(t)u_0(t) + \varphi(t, x, u_0(t)), \quad x(t_0) = x_0$$

is unique on the right maximal interval of existence.

Let $X(t, s)$ be a solution of the matrix problem $X^{\cdot} = A(t)X$, $X(s) = E$, where E is the unit matrix, and let $Y(t, s)$ be a solution of the adjoint matrix problem $Y^{\cdot} = -A^*(t)Y$, $Y(s) = E$. We construct the matrix

$$W(t, s) = \int_s^t Y^*(\tau, s) B(\tau) B^*(\tau) Y(\tau, s)\, d\tau, \quad s \leqslant t$$

It is known (see [4], for example) that system (1.2) is completely controllable on I if and only if $\det W(t_1, t_0) \neq 0$. Therefore, the matrices

$$K(t, s) = \begin{cases} X(t, t_0)W(t_1, t)W^{-1}(t_1, t_0)X(t_0, s), & t_0 \leqslant s \leqslant t \leqslant t_1 \\ -X(t, t_0)W(t, t_0)W^{-1}(t_1, t_0)X(t_0, s), & t_0 \leqslant t < s \leqslant t_1 \end{cases}$$

$$L(t, s) = -B^*(t)Y(t, t_0)W^{-1}(t_1, t_0)Y^*(s, t_0)$$

exist. In addition, we introduce the following notation:

$$k(t) = \max_{s \in I} |K(t, s)|, \qquad l(t) = \max_{s \in I} |L(t, s)| \tag{1.3}$$
$$|P| = \max_{|x| \leqslant 1} |Px|$$

where $|P|$ is the norm of matrix P.

Definition 1. Suppose that the time interval $I = [t_0, t_1]$ has been fixed. System (1.1) is said to be completely controllable if for each $x_0 \in R^n$ we can find a control $u_0(t) \in R^m$, continuous on I, which translates the position (t_0, x_0) of system (1.1) into the position $(t_1, 0)$.

Theorem 1. Assume that we have found functions $a(t)$, $b(t)$, $c(t)$, continuous on I, and numbers $\alpha, \beta \geqslant 0$ ensuring the inequality

$$|\varphi(t, x, u)| \leqslant a(t)|x|^{\alpha} + b(t)|u|^{\beta} + c(t) \tag{1.4}$$

for all $(t, x, u) \in I \times R^n \times R^m$. Assume, further, that at least one of the conditions

1) $\alpha < 1$, $\beta < 1$; 2) $\alpha < 1$, $\beta = 1$, $\int bl\,dt < 1$

3) $\alpha = 1$, $\beta < 1$, $\int ak\,dt < 1$; 4) $\alpha = \beta = 1$, $\int (ak + bl)\,dt < 1$

has been fulfilled. Then system (1.1) is completely controllable. Here and everywhere throughout the following analysis

$$\int f\,dt = \int_{t_0}^{t_1} f(t)\,dt$$

Note. From Theorem 1 it follows, in particular, that system (1.1) is completely controllable under constrained inequalities $(|\varphi| \leqslant c,\ (t, x, u) \in I \times R^n \times R^m)$. This result is contained in [5, 10] as well.

Definition 2. Assume that the time interval $I = [t_0, t_1]$ and a set $\Omega = \Omega(t, x)$ of space R^m have been fixed, where the set Ω is continuous in the sense of the Hausdorff metric and depends upon the point $(t, x) \in I \times S_\gamma$, where $S_\gamma = \{x \in R^n : |x| < \gamma\}$. System (1.1) is said to be locally controllable in-the-small if for any $\varepsilon \in (0, \gamma]$ there exists $\delta > 0$ such that for each $x_0 \in S_\delta$ we can find a control $u_0(t)$, continuous on I, which translates the position (t_0, x_0) of system (1.1) into the position $(t_1, 0)$, and, in addition, (1) the trajectory $x_0(t)$ of system (1.1), corresponding to control $u_0(t)$, satisfies the inclusion $x_0(t) \in S_\varepsilon$; (2) the control $u_0(t)$ and the trajectory $x_0(t)$ are such that $u_0(t) \in \Omega(t, x_0(t))$, $t \in I$.

Theorem 2. Assume the existence of functions $a(t)$, $b(t)$, continuous on I, of numbers $\alpha, \beta \geqslant 0$ and of a neighborhood U of point $x = 0$, $u = 0$ such that the inequality

$$|\varphi(t, x, u)| \leqslant a(t)\,|x|^\alpha + b(t)\,|u|^\beta \tag{1.5}$$

is fulfilled for all $(t, x, u) \in I \times U$. Assume further, that $0 \in \operatorname{int} \Omega(t, 0)$, $t \in I$ and that at least one of the conditions

1) $\alpha = \beta = 1$, $\int (ak + bl)dt < 1$; 2) $\alpha = 1$, $\beta > 1$, $\int ak\,dt < 1$
3) $\alpha > 1$, $\beta = 1$, $\int bl dt < 1$; 4) $\alpha > 1$, $\beta > 1$

is fulfilled. Then system (1.1) is locally controllable in-the-small.

Note. The known theorem on local controllability in the first approximation (see [3], p. 61) follows from Theorem 2 with $\alpha = \beta = 2$.

The following theorem includes the results obtained in [6, 7]. We denote $r(t, \xi) = \max |\varphi(t, x, u)|$, $|x| \leqslant \xi$, $|u| \leqslant \xi$.

Theorem 3. If $\lim \xi^{-1} \int r(t, \xi)dt = 0$, $\xi \to \infty$, then system (1.1) is completely controllable. If $\lim \xi^{-1} \int r(t, \xi)dt = 0$, $\xi \to 0$ and $0 \in \operatorname{int} \Omega(t, 0)$, $t \in I$, then system (1.1) is locally controllable in-the-small.

In a number of cases the investigation of the controllability of system (1.1) can be replaced by an investigation of the controllability of a simpler system. This follows from Theorem 4 presented below.

Let us assume that among the rows $b_1^*, \ldots, b_n^*$ of matrix $B(t)$ there are m rows $b_{i_1}^*, \ldots, b_{i_m}^*$ such that the ($m \times m$)-matrix

$$B_0(t) = \begin{Vmatrix} b_{i_1}^* \\ \cdots \\ b_{i_m}^* \end{Vmatrix}$$

is nonsingular for $t \in I$. We choose the components with indices $i_1, \ldots, i_m$ in vector φ and we set up the vector $\varphi_0(t, x, u) = \operatorname{col}(\varphi_{i_1}, \ldots, \varphi_{i_m})$. We examine the equation

$$z = -B_0^{-1}(t)\,\varphi_0(t, x, z) + u \tag{1.6}$$

relative to the m-vector z.

Theorem 4. If for all $(t, x, u) \in I \times R^n \times R^m$ Eq. (1.6) has at least one solution $z_0(t, x, u)$ depending continuously on point (t, x, u), then the complete controllability of system (1.1) follows from the complete controllability of the system

$$x^{\cdot} = A(t)x + B(t)u + \varphi(t, x, z_0(t, x, u)) - (B(t) B_0^{-1}(t) \times \varphi_0(t, x, z_0(t, x, u)) \quad (1.7)$$

If, however, system (1.7) is locally controllable in-the-small and if the inclusion $z_0(t, x, u) \in \Omega(t, x)$ holds for each point $(t, x) \in I \times S_\gamma$ (where γ is the number occurring in Definition (2)) and for each $u \in \Omega(t, x)$, then system (1.1) is locally controllable in-the-small.

Note. From the Boll-Brauer theorem it follows that when the inequality

$$|\varphi_0(t, x, z)| \leqslant a(t, x)|z|^\beta + b(t, x), \qquad (t, x) \in I \times R^n$$

is satisfied, Eq. (1.6) has a continuous solution $z_0(t, x, u)$ if $\beta < 1$ or if $\beta > 1$ and $b(t, 0) = 0$ (in the second case the solution exists only in a sufficiently small neighborhood of the point $x = 0, u = 0$). The estimate $|z_0(t, x, u)| \leqslant h(t, x, u)$ holds here, where h is the smallest positive root of the equation

$$|B_0^{-1}(t)| \ (a(t, x) h^\beta + b(t, x)) + |u| = h$$

Examples. 1. Assume that the matrix $B(t)$ in system (1.1) is of dimension ($n \times n$) and that $\det B(t) \neq 0$, $t \in I$, while $\varphi(t, x)$ is independent of u. Then when $B_0 = B$, $\varphi_0 = \varphi$, Eq. (1.6) has a unique solution, while system (1.7) coincides with the linear system (1.2).

2. Suppose that the linear stationary system $x^{\cdot} = Ax + Bu$, $u \in R^m$ is completely controllable. Then in the phase space R^n we can choose a basis so that the matrix B can be written as

$$B = \begin{Vmatrix} \Theta \\ E \end{Vmatrix}$$

where Θ is a zero ($(n-m) \times m$)-matrix, E is a unit ($m \times m$)-matrix. Assume that the system being examined is perturbed by $\varphi(t, x, u) = \mathrm{col}(f, \varphi_0)$, where $f(t, x) = \mathrm{col}(\varphi_1, \ldots, \varphi_{n-m})$ is independent of u and $\varphi_0(t, x, u) = \mathrm{col}(\varphi_{n-m+1}, \ldots, \varphi_n)$ satisfies the estimate

$$|\varphi_0(t, x, u)| \leqslant a(t, x)|u|^\beta + b(t, x), \ \beta < 1, \ (t, x) \in I \times R^n$$

Then, by virtue of Theorem 4, the system

$$x^{\cdot} = Ax + Bu + \varphi(t, x, u) \qquad (1.8)$$

is completely controllable if the system $x^{\cdot} = Ax + Bu + \psi(t, x)$, where $\psi = \mathrm{col}(f, 0)$ is completely controllable. Thus, the property of complete controllability of system (1.8) is invariant relative to the components $\varphi_{n-m+1}, \ldots, \varphi_n$ of perturbation φ. Therefore, for example, the system equivalent to $x^{(n)} = f(t, x, x', \ldots, x^{n-1}) + u$, is completely controllable on any time interval, independently of the form of f.

2. Proofs. The proofs of the first three theorems use the Schauder principle on the existence of a fixed point. We note that [8, 9] also were devoted to the application of the Schauder principle to the investigation of the controllability of a nonlinear system.

Lemma 1. Let $x_0(t)$; $u_0(t)$ be the solution of system

$$x(t) = \int K(t, s)\varphi(s, x(s), u(s))ds + K(t, t_0)x_0 \qquad (2.1)$$
$$u(t) = \int L(t, s)\varphi(s, x(s), u(s))ds + L(t, t_0)x_0$$

Then, control $u = u_0(t)$ translates the position (t_0, x_0) of system (1.1) into the position $(t_1, 0)$ along trajectory $x_0(t)$.

This lemma can be proved by a direct check.

Lemma 2. System (2.1) has at least one solution for any $x_0 \in R^n$ under the hypotheses of Theorem 1.

Proof. We write system (2.1) as $y = Fy$, where

$$y = \left\|\begin{matrix} x \\ u \end{matrix}\right\|, \quad Fy = \int R(t, s)\varphi(s, y(s))ds + R(t, t_0)x_0, \quad R(t, s) = \left\|\begin{matrix} K(t, s) \\ L(t, s) \end{matrix}\right\|$$

From the structure of matrices K and L it follows that the ($(n+m) \times n$)-matrix $R(t, s)$ satisfies the Radon conditions for $(t, s) \in I \times I$. Further, since φ is continuous in the collection of arguments $(t, y) \in I \times R^{n+m}$, by virtue of a known theorem of Nemytskii the operator F is completely continuous as an operator acting from $C = C(I, R^{n+m})$ into C, where C is a Banach space of functions $y(t)$, continuous on I, with values in R^{n+m} and norm $\|y\| = \max|y(t)|$, $t \in I$.

Let inequality (1.4) be fulfilled for $\alpha < 1$, $\beta < 1$. We consider the set $C_1 \subset C$ of functions $y = \mathrm{col}(x, u)$ satisfying the conditions $|x(t)| \leqslant \xi_0$, $|u(t)| \leqslant \xi_0$, where ξ_0 is a positive solution of the equation

$$\xi = M(a_1\xi^\alpha + b_1\xi^\beta + c_1 + |x_0|)$$
$$M = \max\{\max k(t), \max l(t), t \in I\}, \quad a_1 = \int a dt, \ b_1 = \int b dt, \ c_1 = \int c dt$$

$k(t)$, $l(t)$ are defined by equalities (1.3). Set C_1 is a bounded closed convex set in space C. Let us show that set C_1 is invariant relative to mapping F. In fact, let $y = \mathrm{col}(x, u) \in C_1$, $Fy = \mathrm{col}(x_1, u_1)$. For x_1 we have the estimate

$$|x_1(t)| \leqslant \int |K(t, s)||\varphi(s, x(s), u(s))|ds + |K(t, t_0)||x_0| \leqslant$$
$$M\left(\int(a(t)|x(t)|^\alpha + b(t)|u(t)^\beta + c(t))dt + |x_0|\right) \leqslant M(a_1\xi_0^\alpha + b_1\xi_0^\beta +$$
$$c_1 + |x_0|) = \xi_0$$

Analogously, $|u_1(t)| \leqslant \xi_0$. Consequently, $Fy \in C_1$. System (2.1) has a solution by virtue of the Schauder principle.

Now suppose that inequality (1.4) is fulfilled for $\alpha < 1$, $\beta = 1$. By ξ_0 we denote a positive root of the equation

$$\xi = k_1/\Delta_1(a_1\xi^\alpha + c_1 + |x_0|)$$
$$k_1 = \max_{t \in I} k(t), \quad \Delta_1 = 1 - \int bl dt > 0, \quad a_1 = \int a dt, \quad c_1 = \int c dt$$

The existence of ξ_0 follows from the conditions $\alpha > 1$, $\Delta_1 > 0$ of Theorem 1. It is clear, therefore, that the system of equations

$$\xi = k_1\left(a_1\xi^\alpha + \int b(t)\psi(t)dt + c_1 + |x_0|\right), \quad \psi(t) = \frac{l(t)}{k_1}\xi$$

has a positive solution ξ_0, $\psi_0(t) = (l(t)/k_1)\xi_0$.

We construct the set C_2 of functions $y = \operatorname{col}(x, u)$, continuous on I, satisfying the inequalities $|x(t)| \leqslant \xi_0$, $|u(t)| \leqslant \psi_0(t)$. Set C_2 is a bounded closed convex set in space C and, in addition, C_2 is invariant relative to mapping F. This is verified by the usual method

$$y = \operatorname{col}(x, u) \in C_2, \qquad Fy = \operatorname{col}(x_1, u_1)$$

$$|x_1(t)| \leqslant k_1 (\int (a(t) |x(t)|^\alpha + b(t) |u(t)| + c(t) \, dt + |x_0|) \leqslant$$
$$k_1 (a_1 \xi_0^\alpha + \int b(t) \psi_0(t) \, dt + c_1 + |x_0|) = \xi_0.$$

Analogously, $|u_1(t)| \leqslant \psi_0(t)$. Consequently, $Fy \in C_2$. System (2.1) has a solution by virtue of the Schauder principle.

The proof is analogous to the one just considered when $\alpha = 1$, $\beta < 1$.

Let inequality (1.4) be fulfilled for $\alpha = \beta = 1$. Let us examine the linear system of integral equations in the scalar functions $\xi(t)$, $\psi(t)$

$$\begin{aligned} \xi(t) &= k(t) (\int (a(t) \xi(t) + b(t) \psi(t)) \, dt + |x_0| + c_1) \\ \psi(t) &= l(t) (\int (a(t) \xi(t) + b(t) \psi(t)) \, dt + |x_0| + c_1) \end{aligned} \tag{2.2}$$

It turns out that system (2.2) has the positive solution

$$\xi_0(t) = \frac{k(t)}{\Delta} (|x_0| + c_1), \qquad \psi_0(t) = \frac{l(t)}{\Delta} (|x_0| + c_1) \tag{2.3}$$
$$\Delta = 1 - \int (ak + bl) \, dt > 0$$

We construct a set C_4 of functions $y = \operatorname{col}(x, u)$, continuous on I, satisfying the inequalities

$$|x(t)| \leqslant \xi_0(t), \qquad |u(t)| \leqslant \psi_0(t)$$

Set C_4 is a bounded closed convex set in space C. In addition, we can verify that C_4 is invariant relative to mapping F. A reference to the Schauder principle completes the proof of Lemma 2.

For further analysis we shall need the following estimate of the solution $x_0(t)$, $u_0(t)$ of system (2.1), perceivable directly from (2.3) when $\alpha = \beta = 1$:

$$|x_0(t)| \leqslant \frac{k(t)}{\Delta} (|x_0| + c_1), \quad |u_0(t)| \leqslant \frac{l(t)}{\Delta} (|x_0| + c_1), \quad t \in I \tag{2.4}$$

The proof of Theorem 1 follows from Lemmas 1 and 2.

Lemma 3. Let the hypotheses of Theorem 2 be fulfilled (except for the condition $0 \in \operatorname{int} \Omega(t, 0)$, $t \in I$). Then we can find $\eta > 0$ such that for each $x_0 \in S_\eta = \{x \in R^n : |x| < \eta\}$ the system (2.1) has at least one solution $x_0(t)$, $u_0(t)$ and $|x_0(t)| + |u_0(t)| \to 0$ as $|x_0| \to 0$ uniformly with respect to $t \in I$.

Proof. We choose a number $\nu > 0$ so small that $S_\nu \subset U$, where $S_\nu = \{y \in R^{n+m}: |y| < \nu\}$, and U is the region in which inequality (1.5) is fulfilled. From the function φ we construct the new function

$$\varphi_1(t, x, u) = \varphi_1(t, y) + \begin{cases} \varphi(t, y), & t \in I, \quad y \in S_\nu \\ \varphi\left(t, \dfrac{\nu y}{|y|}\right), & t \in I, \quad y \notin S_\nu \end{cases}$$

It is clear that the function φ_1 is continuous in all arguments and satisfies inequality (1.5) for all $(t, x, u) \in I \times R^n \times R^m$

Let us assume temporarily that Lemma 3 has been proved for system (2.1) with the function φ_1 instead of φ. Consequently, we have proved that there exists a solution $x_0(t), u_0(t)$ satisfying the condition $|x_0(t)| + |u_0(t)| \to 0$, $|x_0| \to 0$. This signifies that if the number $|x_0|$ is sufficiently small, $y_0(t) = \mathrm{col}(x_0(t), u_0(t)) \in S_\nu$ for all $t \in I$. But, by virtue of the construction of φ_1 we have the identity $\varphi_1(t, y_0(t)) = \varphi(t, y_0(t))$ and, consequently, system (2.1) with the function φ also has a solution. Thus, the existence of a solution of system (2.1) with function φ_1 implies the existence of a solution (for sufficiently small $|x_0|$) of system (2.1) with function φ. Therefore, without loss of generality we assume that inequality (1.5) holds for all $(t, x, u) \in I \times R^n \times R^m$.

We first analyze the case $\alpha = \beta = 1$. By arguing in the same way as in the proof of Lemma 2 we conclude that the solution $x_0(t)$, $u_0(t)$ of system (2.1) exists and that this solution satisfies estimate (2.4) with $c_1 = 0$. Consequently,

$$|x_0(t)| + |u_0(t)| \leqslant \frac{2M}{\Delta}|x_0| \to 0, \qquad |x_0| \to 0$$

$$M = \max\{\max k(t), \ \max l(t), \ t \in I\}$$

uniformly in $t \in I$.

Let inequality (1.5) be fulfilled for $\alpha = 1, \beta > 1$. At first we examine the equation

$$\psi = \frac{l_1}{\Delta_1}(b_1\psi^\beta + |x_0|) \tag{2.5}$$

$$l_1 = \max_{t\in I} l(t), \qquad \Delta_1 = 1 - \int akdt > 0, \qquad b_1 = \int bdt$$

in the unknown ψ. Since $\beta > 1$, Eq. (2.5) has two positive solutions for sufficiently small $|x_0|$, the smallest of which is $\psi_0(x_0)$. It is clear that $\psi_0(x_0) \to 0$, $|x_0| \to 0$. We now consider a system of equations in $\xi(t)$, ψ

$$\xi(t) = \frac{k(t)}{l_1}\psi, \qquad \psi = l_1\left(\int a(t)\,\xi(t)\,dt + b_1\psi^\beta + |x_0|\right) \tag{2.6}$$

Under the existence conditions for the solution of Eq. (2.5), system (2.6) has the positive solution $\xi_0(t) = (k(t)/l_1)\psi_0$, $\psi = \psi_0$ and, consequently, $\xi_0(t) + \psi_0 \to 0$, $|x_0| \to 0$ uniformly in $t \in I$.

By C_2 we denote the set of functions $y(t) = \mathrm{col}(x(t), u(t))$, continuous on I, satisfying the inequalities $|x(t)| \leqslant \xi_0(t)$, $|u(t)| \leqslant \psi_0$. We can show that the set C_2 is invariant relative to the mapping F constructed in the proof of Lemma 2. By virtue of the Schauder principle system (2.1) has a solution $x_0(t)$, $u_0(t)$ satisfying the estimate

$$|x_0(t)| + |u_0(t)| \leqslant \xi_0(t) + \psi_0$$

The arguments for the case $\alpha > 1$, $\beta = 1$ are analogous to those just presented.

Suppose that inequality (1.5) is fulfilled for $\alpha > 1$, $\beta > 1$. We denote the smallest positive solution of the equation

$$\psi = M(a_1\psi^\alpha + b_1\psi^\beta + |x_0|)$$

$$M = \max\{\max k(t), \ \max l(t), \ t \in I\}, \qquad a_1 = \int adt, \qquad b_1 = \int bdt$$

by $\psi_0(x_0)$. The solution $\psi_0(x_0)$ exists for sufficiently small $|x_0|$ and $\psi_0(x_0) \to 0$ as $|x_0| \to 0$. It turns out that the set $C_4 \subset C$ of functions $y(t) = \mathrm{col}(x(t), u(t))$, continuous on I, satisfying the inequalities $|x(t)| \leqslant \psi_0$, $|u(t)| \leqslant \psi_0$, is invariant relative to mapping F. This information suffices to complete the proof of Lemma 3.

Proof of Theorem 2. The condition $0 \in \operatorname{int} \Omega(t, 0)$ together with the continuity of set $\Omega(t, x)$ in (t, x) signifies that we can find numbers $\delta_1 > 0$, $\varepsilon_1 \in (0, \gamma]$ (where γ is the number occurring in Definition 2) ensuring the inclusion $S_{\delta_1} \subset \operatorname{int} \Omega(t, x)$ for all $(t, x) \in I \times S_{\varepsilon_1}$. Suppose that the ε occurring in Definition 2 is given. We construct $\varepsilon_2 = \min\{\varepsilon_1, \varepsilon\}$. From the numbers ε_2 and δ_1 we select δ so small that, first, $\delta < \delta_1$ and, second, for all $x_0 \in S_\delta$ the solution $x_0(t)$, $u_0(t)$ of system (2.1) satisfies the inclusions $x_0(t) \in S_{\varepsilon_2}$, $u_0(t) \in S_{\delta_1}$, $t \in I$. This can be achieved by virtue of Lemma 3.

From Lemma 1 it follows that control $u_0(t)$ translates the position (t_0, x_0) of system (1.1) into the position $(t_1, 0)$ along trajectory $x_0(t)$. In addition, $x_0(t) \in S_{\varepsilon_2} \subset S_\varepsilon$. Further, the inclusion $u_0(t) \in \operatorname{int} \Omega(t, x)$ for all $(t, x) \in I \times S_\varepsilon$, follows from the inclusion $u_0 \ (t) \in S_{\delta_1}$, but $x_0(t) \in S_{\varepsilon_1}$ and, therefore, $u_0(t) \in \operatorname{int} \Omega(t, x_0(t)) \subset \Omega(t, x_0(t))$. Theorem 2 is proved.

Proof of Theorem 3. If

$$\lim_{\xi\to\infty} \xi^{-1} \int r(t, \xi)\, dt = 0$$

then the equation

$$M\left(\int r(t, \xi) dt + |x_0|\right) = \xi \tag{2.7}$$

$$M = \max\{\max k(t),\ \max l(t),\ t \in I\}$$

has a positive solution ξ_0 for each x_0. We can verify that the bounded closed convex set $C_\xi \subset C$ of functions $y(t) = \operatorname{col}(x(t), u(t))$, continuous on I and satisfying the inequalities $|x(t)| \leqslant \xi_0$, $|u(t)| \leqslant \xi_0$, is invariant relative to the mapping F constructed in the proof of Lemma 2. If

$$\lim \xi^{-1} \int r(t, \xi) dt = 0, \quad \xi \to 0$$

then Eq. (2.7) has the positive solution $\xi_0 = \xi_0(x_0) \to 0$ as $|x_0| \to 0$. In this case the same set $|x(t)| \leqslant \xi_0$, $|u(t)| \leqslant \xi_0$ serves as the set invariant relative to F. The data presented suffices to complete the proof of Theorem 3.

Proof of Theorem 4. Let $u_1(t)$ be the control translating the position (t_0, x_0) of system (1.7) into the position $(t_1, 0)$ along trajectory $x_1(t)$. Then by direct verification we are convinced that the control $u_0(t) = z_0(t, x_1(t), u_1(t))$ translates the position (t_0, x_0) of system (1.1) into the position $(t_1, 0)$ along the trajectory $x_0(t) = x_1(t)$, $t \in I$. Therefore, the complete controllability of system (1.1) follows from the complete controllability of system (1.7).

If, however, system (1.7) is locally controllable in-the-small, then $u_1(t) \in \Omega(t, x_1(t))$. By virtue of the condition $z_0(t, x, u) \in \Omega(t, x)$ for $u \in \Omega(t, x)$, we have the inclusion

$$u_0(t) = z_0(t, x_1(t), u_1(t)) \in \Omega(t, x_1(t))$$

so that system (1.1) also is locally controllable in-the-small.

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STABILITY ANALYSIS OF DYNAMIC SYSTEMS WITH COUPLINGS AND INTEGRALS OF MOTIONS

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We give a method for obtaining the stability conditions for nonlinear systems, based on an analysis of the linearized coupling equations and of the linearized or quadratic expressions for the integrals of motion. Liapunov's method is usually employed in the investigation of the stability of dynamic systems. The investigation of the Hamiltonian function is a convenient tool for systems with internal energy dissipation. In fact, in the development of the Thompson (Lord Kelvin)-Tait-Chetaev theorem [1 – 4] it was shown that the positive definiteness of the Hamiltonian function provides the necessary and sufficient stability conditions in the case of complete dissipation. We have obtained just sufficient conditions for system with partial dissipation; moreover, the method does not yield the possibility of obtaining far-reaching inferences on stability on the basis of the analysis of the linearized equations. It should be noted also that in several cases it is convenient to introduce a number of variables, exceeding the number of degrees of freedom, and to examine the couplings. Then the equations can be simplified or represented in a form convenient for stability analysis.

We can derive the equations of motion of the system under consideration with the aid